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LOCALIZATION AND FINITE SIMPLE GROUPS

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ABSTRACT

A homomorphism from a group H to a group G is a localization if and only if it induces a bijection between $\mathrm{Hom}(G,G)$ and $\mathrm{Hom}(H,G)$. In this paper we study the equivalence relation that localization induces on the family of finite non-abelian simple groups.

1. Introduction

Following [20] we define a group homomorphism ϕ: $H \to G$ to be a **localization** if and only if ϕ induces a bijection between $\mathrm{Hom}(G,G)$ and $\mathrm{Hom}(H,G)$. Let $\mathcal{L}$ denote the family of finite non-abelian simple groups, one of each isomorphism type. We are interested in the equivalence relation localization induces on $\mathcal{L}$. This was first investigated in [20, Theorem] where the authors establish that all the simple alternating groups, the sporadic simple groups except the Monster, the simple groups $\mathrm{PSL}_2(q)$ and $\mathrm{PSU}_3(q)$ are all in the same equivalence class. They call this equivalence class the **rigid component** of $\mathcal{L}$ and denote it by $\mathcal{L}_0$. In [20] it is suggested that perhaps $\mathcal{L} = \mathcal{L}_0$ and that in any case $\mathcal{L}_0$ should contain the majority of the simple groups. This latter suggestion is part of what

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we establish in this note. We say that a simple group is **isolated** in $\mathcal{L}$ if it is the only member of its equivalence class.

THEOREM 1.1: *The rigid component $\mathcal{L}_0$ of $\mathcal{L}$ contains all the non-abelian simple groups except $\mathrm{PSp}_4(p^{2^c})$ where p is an odd prime and $c > 0$. Furthermore, the groups not contained in $\mathcal{L}_0$ are isolated in $\mathcal{L}$.*

In [20] a group theoretic criterion is given which is equivalent to the existence of a localization of a simple group H into a simple group G.

THEOREM 1.2 ([20]): *Suppose that H and G are non-abelian finite simple groups and i: $H \to G$ is inclusion. Then i is a localization if and only if the following three conditions are satisfied:*

(1) *The inclusion i extends to an inclusion i: $\mathrm{Aut}(H) \to \mathrm{Aut}(G)$.*
(2) *Any subgroup of G which is isomorphic to H is conjugate to $i(H)$ in $\mathrm{Aut}(G)$.*
(3) $C_{\mathrm{Aut}(G)}(i(H)) = 1$.

If there is a non-trivial homomorphism from a simple group H to a group G, then it is of course a monomorphism. Thus if there is a localization from a simple group H to a group G, then H can be identified with a subgroup of G. If the three conditions of Theorem 1.2 are satisfied, then we shall say that H is **localized** in G. Also in [20] an elementary criterion for an inclusion of H into the alternating group of degree n to be a localization is given:

THEOREM 1.3 ([20]): *Let H be a non-abelian finite simple group and K be a maximal subgroup of H of index $n \geq 7$. Suppose that the following conditions are satisfied:*

(1) *the order of K is maximal; and*
(2) *any subgroup of H of index n is conjugate to K.*

Then the permutation representation which includes H into $\mathrm{Alt}(n)$ is a localization.

In Proposition 2.6 below, we obtain a result similar to Theorem 1.3 which gives a representation-theoretic criterion on a simple group H which guarantees that it is localized in either $\mathrm{P}\Omega_n^{\epsilon}(r)$ or $\mathrm{PSp}_n(r)$ for some prime r and integer n.

In Section 2 we prove that, with the exception of $\mathrm{PSp}_4(p^a)$ with p odd, all the non-abelian simple groups are contained in the rigid component $\mathcal{L}_0$. In Section 3 we investigate the localizations of $\mathrm{PSp}_4(p^a)$, the aim being to prove that the groups $\mathrm{PSp}_4(p^{2^c})$ with $c > 0$ are isolated in $\mathcal{L}$ and that all the other groups are contained in $\mathcal{L}_0$. Our notation for the simple groups is consistent with Kleidman and Liebeck [10].

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2. The rigid component

In this section we show that all the simple groups apart from $\mathrm{PSp}_4(p^a)$ with p odd are contained in the rigid component of $\mathcal{L}$. We take as our starting point the results from [20]. So we assume that $\mathcal{L}_0$ contains all the simple alternating groups, the simple groups $\mathrm{PSL}_2(q)$, $\mathrm{PSU}_3(q)$ and all the sporadic simple groups except the Monster.

LEMMA 2.1: *Suppose that H is a simple group of Lie type defined in characteristic p and P is a parabolic subgroup of H. Assume that any parabolic subgroup of H which has the same order as P is conjugate to P. If $|P|$ is maximal among the orders of proper subgroups of H, then $H \in \mathcal{L}_0$.*

Proof: Suppose that $P^* \leq H$ and $|P^*| = |P|$. Then by Sylow's Theorem we may suppose that P^* contains a Sylow p-subgroup of H. Therefore, a result of Borel and Tits [6, Theorem 2.6.7] implies that P^* is contained in a parabolic subgroup $\hat{P}$ of H. The maximality of $|P|$ means that $P^* = \hat{P}$ and then P is conjugate to P^* by assumption. Thus, as all the alternating groups are in $\mathcal{L}_0$, Theorem 1.3 implies $H \in \mathcal{L}_0$ and completes the proof. ∎

PROPOSITION 2.2: *Suppose that p is a prime and $q = p^a$. Then the following simple groups are contained in $\mathcal{L}_0$.*

(1) $\mathrm{PSp}_{2n}(q)$, $n > 2$.
(2) $\mathrm{PSU}_n(q)$, $n \geq 4$.
(3) $\mathrm{P\Omega}^{\epsilon}_{2n}(q)$, $n \geq 4$, $\epsilon = \pm$ *and* $(n, \epsilon) \neq (4, +)$.
(4) $\mathrm{P\Omega}_{2n+1}(q)$, $n \geq 3$ *and* q *odd.*

Proof: Let G be any of the groups listed in the statement of the proposition. The minimal permutation degree d for G is presented in [4] (see [10, Table 5.2A] for a slightly corrected list). If the point stabilizer in the minimal permutation representation of G is a parabolic subgroup of G, then by examining the orders of the parabolic subgroups and then using Lemma 2.1 we have G localized in $\mathrm{Alt}(d)$ (this is the point in the proof where groups like $\mathrm{PSL}_n(q)$, $\mathrm{P}\Omega_8^+(q)$, $\mathrm{F}_4(q)$ and $\mathrm{PSp}_4(q)$ have to be excluded). According to [10, Table 5.2] (together with two further improvements), this leaves the groups $\mathrm{PSp}_{2n}(2)$ with $n \geq 3$, $\mathrm{P}\Omega_{2n+1}(3)$ with $n \geq 3$, $\mathrm{P}\Omega_{2n}^+(2)$ and $\mathrm{P}\Omega_{2n}^+(3)$ with $n > 4$, $\mathrm{PSU}_n(3)$ with n divisible by 6 and $\mathrm{PSU}_n(2)$ with n even to be considered further. For these groups, we invoke the main result of [11] to see that the point stabiliser in the degree d permutation representation is determined uniquely up to conjugacy by its order in G. An application of Theorem 1.3 completes the proof of Proposition 2.2. ∎

PROPOSITION 2.3: *The following groups are contained in* $\mathcal{L}_0$.

(1) ${}^2\mathrm{B}_2(2^a)$, $a > 1$.
(2) ${}^2\mathrm{G}_2(3^a)$, $a > 1$.
(3) ${}^2\mathrm{F}_4(2^a)$, $a > 1$.

Proof: We examine the maximal subgroups of each of the groups and then apply Theorem 1.3. See, for example, [6, Theorems 6.5.4 and 6.5.6] for ${}^2\mathrm{B}_2(2^a)$ and ${}^2\mathrm{G}_2(3^a)$, $a > 1$ and [17] for ${}^2\mathrm{F}_4(2^a)$, $a > 1$. Alternatively use the main result in [12]. ∎

We recall that for a group H and a field k, two representations $\rho_1\colon H \to \mathrm{GL}_n(k)$ and $\rho_2\colon H \to \mathrm{GL}_n(k)$ are **quasiequivalent** if there is $\theta \in \mathrm{Aut}(H)$ such that $\theta\rho_1$ is equivalent to ρ_2.

LEMMA 2.4: *Suppose that k is a field, H is a finite group and $\phi\colon H \to G = \mathrm{GL}_n(k)$ is a faithful representation. Then the number of inequivalent representations of H which are quasiequivalent to ϕ is* $\frac{|\mathrm{Aut}(H)|}{|N_G(\phi(H))/C_G(\phi(H))|}$.

Proof: For $g \in G$ let c_g represent the automorphism of G induced by conjugation. Then, for $g \in N_G(\phi(H))$ the map $g \mapsto \phi.c_g.\phi^{-1}$ embeds $N_G(\phi(H))/C_G(\phi(H))$ into $\mathrm{Aut}(H)$. Suppose that $\alpha, \beta \in \mathrm{Aut}(H)$. Then $\alpha.\phi$ is equivalent to $\beta.\phi$ if and only if there is a $g \in G$ such that $\alpha.\phi = \beta.\phi.c_g$ which is if and only if $\beta^{-1}.\alpha = \phi.c_g.\phi^{-1}$. Thus the number of inequivalent representations of H which are quasiequivalent to ϕ is equal to the number of cosets of the image of $N_G(\phi(H))$ in $\mathrm{Aut}(H)$ as claimed. ∎

LEMMA 2.5: *Suppose that G acts absolutely irreducibly on the kG-module V by the representation ϕ. Assume that G leaves invariant a symplectic, quadratic or* *a unitary form f, and let $h \in \mathrm{GL}(V)$. If $\phi(G)^h$ preserves f, then h is contained in the similarity group of f.*

Proof: Define $f_h(v,w) = f(vh,wh)$ for all $v,w \in V$. Then f_h has the same type as f. Furthermore, for all $v,w \in V$ and $g \in G$, $f_h(vg,wg) = f(vgh,wgh) = f(vhh^{-1}gh, vhh^{-1}gh) = f(vhg^h, whg^h) = f(vh,wh) = f_h(v,w)$. Therefore, f_h is also G-invariant. It follows from [10, 2.10.3] that $f_h = \lambda f$ for some $\lambda \in k$ and this means precisely that h is in the similarity group of f. ∎

PROPOSITION 2.6: *Suppose that G is a simple group with universal covering group $\hat{G}$ and r is a prime with $r \equiv 1 \pmod{|\hat{G}|}$. If $\hat{G}$ has a unique non-trivial complex representation ϕ of minimal degree d, then G is localized in one of $\mathrm{PSp}_d(r)$ or $\mathrm{P}\Omega_d^\epsilon(r)$.*

Proof: Let ϕ be the unique non-trivial complex representation of $\hat{G}$ of minimal degree d. Note that as r and $|\hat{G}|$ are coprime, this representation can be realized over a field of characteristic r. Then, as $r \equiv 1 \pmod{|\hat{G}|}$, ϕ can be realized over $\mathrm{GF}(r)$. Thus we may take $\phi\colon \hat{G} \to \mathrm{GL}_d(r)$. Since ϕ is unique of this degree, ϕ and the dual of ϕ are equivalent. Hence, setting $H = \phi(\hat{G})$, we see that H preserves either a non-degenerate quadratic form or a non-degenerate symplectic form. Thus ϕ can be considered as a homomorphism from $\hat{G}$ into one of $\mathrm{Sp}_d(r)$ or $\Omega_d^\epsilon(r)$. By Lemmas 2.4 and 2.5, $N_{\mathrm{GL}_d(r)}(H)/C_{\mathrm{GL}_d(r)}(H)H \cong \mathrm{Out}(\hat{G})$ and $N_{\mathrm{GL}_d(r)}(H) \leq \mathrm{GSp}_d(r)$ or $\mathrm{GO}_d^\epsilon(r)$. By the choice of r, ϕ is absolutely irreducible, so $C_{\mathrm{GL}_d(r)}(H) = Z(\mathrm{GL}_d(r))$. Hence $\mathrm{Aut}(\hat{G})$ embeds into $\mathrm{PGSp}_d(r)$ or into $\mathrm{PGO}_d^\epsilon(r)$. Finally, the uniqueness and minimality of ϕ together with Lemma 2.5 imply that the conjugacy class of $H/Z(H)$ is the unique conjugacy class of subgroups isomorphic to $H/Z(H)$ contained in $\mathrm{PGSp}_d(r)$ or in $\mathrm{PGO}_d^\epsilon(r)$. The proposition now follows from Theorem 1.2. ∎

The following lemma is mentioned by Robert Wilson in his review of [20] for Zentralblatt.

LEMMA 2.7: *The Monster is in $\mathcal{L}_0$.*

Proof: This is immediate from Proposition 2.6 as the Monster has a unique minimal complex representation (of degree 196883). ∎

PROPOSITION 2.8: *Suppose that p is a prime and $q = p^a$. Then the following simple groups are contained in $\mathcal{L}_0$.*

(1) *$\mathrm{PSL}_n(q)$ with $n \geq 3$ and $(n,q) \neq (3,2),(3,4),(4,2)$ and $(4,3)$.*

(2) $\mathrm{P}\Omega_8^+(q)$ *with* $q \geq 4$.
(3) $\mathrm{PSp}_4(2^a)$ *with* $a > 1$.
(4) $\mathrm{G}_2(q)$ *with* $q > 2$.
(5) ${}^3\mathrm{D}_4(q)$.
(6) $\mathrm{F}_4(q)$.
(7) $\mathrm{E}_n(q)$ *with* $n = 6, 7, 8$ *and* ${}^2\mathrm{E}_6(q)$.

Proof: Let G be one of the groups listed in the proposition and let $\hat{G}$ be its universal covering group. By [15] and [21], under the conditions stipulated, G has a unique projective representation over $\mathbb{C}$ of minimal degree. (Here we note that exceptional covers of G have been taken into account in [15] and [21].) Furthermore, [21, Proposition 7.2 and Theorem 7.6] indicates that the minimal projective representation for $\mathrm{P}\Omega_8^+(q)$, $q \geq 5$ is unipotent. This means that it extends uniquely to a representation of the covering group of $\mathrm{Spin}_8(q)$ with the centre of the latter group in the kernel of the representation (see [2, page 380]). Noting that for the other classes of groups the universal covering group of G has cyclic centre, we see $\hat{G}$ has a unique complex representation of minimal degree. Thus the result follows from Proposition 2.6. ∎

PROPOSITION 2.9: *Any simple group other than* $\mathrm{PSp}_4(p^a)$ *with* p *odd is in* $\mathcal{L}_0$.

Proof: Using Propositions 2.2, 2.3 and 2.8 with Lemma 2.7 and [20] we only need to consider those groups explicitly excluded from consideration in Propositions 2.3 and 2.8. However, we note that $\mathrm{PSL}_3(2) \cong \mathrm{PSL}_2(7)$ and $\mathrm{PSL}_4(2) \cong \mathrm{Alt}(8)$ and so these two groups are in $\mathcal{L}_0$. We have $\mathrm{PSL}_4(3) \in \mathcal{L}_0$ by [20] and the embedding $\mathrm{Alt}(6)$ in $\mathrm{PSL}_3(4)$ implies $\mathrm{PSL}_3(4) \in \mathcal{L}_0$ (see [5]). Noting that the group $2.\mathrm{O}_8^+(2)$ is the Weyl group of type E_8, we have embeddings $\mathrm{P}\Omega_8^+(2)$ into $\mathrm{P}\Omega_8^+(3)$ and into $\mathrm{P}\Omega_8^+(5)$. Using [9, Proposition 2.3.8] we see that both of these are localizations and as $\mathrm{P}\Omega_8^+(5) \in \mathcal{L}_0$ by Proposition 2.8, we have both $\mathrm{P}\Omega_8^+(2)$ and $\mathrm{P}\Omega_8^+(3) \in \mathcal{L}_0$. The symplectic group omitted in Proposition 2.8 is $\mathrm{PSp}_4(2)' \cong \mathrm{Alt}(6)$ and the omitted group of type $\mathrm{G}_2(q)$ is $\mathrm{G}_2(2)' \cong \mathrm{PSU}_3(3)$ and so these groups are in $\mathcal{L}_0$. Finally, by [20], we have ${}^2\mathrm{G}_2(3)' \cong \mathrm{PSL}_2(8) \in \mathcal{L}_0$ and ${}^2\mathrm{F}_4(2)' \in \mathcal{L}_0$ via the embedding ${}^2\mathrm{F}_4(2) \to \mathrm{Ru} \to \mathrm{Alt}(4060)$. ∎

3. The groups $\mathrm{PSp}_4(q)$ and isolated groups

The objective of this section is to prove:

PROPOSITION 3.1: *Suppose that p is an odd prime and $q = p^a$ for a positive integer a. Set $H \cong \mathrm{PSp}_4(p^a)$.*

(1) *If $a \neq 2^c$ with $c > 0$, then $H \in \mathcal{L}_0$.*

(2) *If $a = 2^c$ with $c > 0$, then H is isolated in $\mathcal{L}$.*

We investigate the localizations related to $\mathrm{PSp}_4(p^a)$ via a sequence of lemmas. For the rest of this paper we shall assume that p is an odd prime and $q = p^a$.

We begin by showing that if $q = p^{2^c}$ with $c > 0$, then no proper non-abelian simple subgroup of $G = \mathrm{PSp}_4(q)$ is localized in G.

LEMMA 3.2: *Suppose that H is a non-abelian simple group and $G \cong \mathrm{PSp}_4(q)$ with $q = p^{2^c}$, p an odd prime and $c > 0$. If H is localized in G, then $H \cong G$.*

Proof: Suppose that H is a non-abelian simple group which is localized in $G \cong \mathrm{PSp}_4(q)$. By Theorem 1.2, we require that $C_{\mathrm{Aut}(G)}(H) = 1$. So, in particular, we require $C_G(H) = 1$. Since H is contained in a maximal subgroup of G, we can use the list of maximal subgroups of G as presented in [18]. From the structure of the parabolic subgroups of G, we infer that H is not contained in a parabolic subgroup of G. If H is contained in $\mathrm{PSp}_2(q^2)$, then, as q is odd, H is centralized by the diagonal automorphism of G (see [6, Table 4.5.1]), and since the image of $\mathrm{GU}_2(q)$ in $\mathrm{PSp}_4(q)$ has non-trivial centralizer, H is not contained in this subgroup either. The subgroup $(\mathrm{SL}_2(q) \circ \mathrm{SL}_2(q)).2$ is the centralizer of an involution in G, so H is not contained in this subgroup. Next H does not lie in the subgroup $\mathrm{GL}_2(q) : 2$ since this group also has a non-trivial centralizer in G. Since the subfield subgroups are centralized by field automorphisms of G, H is not contained in such a subgroup. If H is a subgroup of $\mathrm{PSL}_2(q)$ arising from an irreducible embedding of $\mathrm{SL}_2(q)$ into $\mathrm{Sp}_4(q)$ (which occurs for $p \geq 5$ and $q \geq 7$), then as we have assumed that H is localized in G, H must also be conjugate to a subgroup of the parabolic subgroup $q^3 : \mathrm{PSL}_2(q) : (q-1)$, which is a contradiction. Since the other types of maximal subgroups of G only occur for q a prime, this shows that H is not contained in any of the maximal subgroups of G and hence $H \cong G$ as claimed. ∎

Suppose that $k = \mathrm{GF}(q^b)$ for some positive integer b and odd prime power q and k' is the subfield of k isomorphic to $\mathrm{GF}(q)$. Assume that V is a vector space of dimension n over k and f is a non-degenerate symplectic form on V. Let $\mathrm{GSp}_n(k)$ be the group of k-similarities of the symplectic space (V, f). Define $\mathrm{GSp}_{n,k'}(k)$ to be the subgroup of $\mathrm{GSp}_n(k)$ which consists of k'-similarities of (V, f); those similarities which satisfy $f(vg, wg) = \lambda f(v, w)$ for some $\lambda \in k'$. Let $T^k_{k'} : k \to k'$ be the trace map. Consider V as a vector space over k' and let $f_{k'}$ be the non-degenerate symplectic form $T^k_{k'}f$. Since $f_{k'}$ is preserved up to similarity by $\mathrm{GSp}_{n,k'}(k)$, $H = \mathrm{GSp}_{n,k'}(k)$ naturally embeds into $G = \mathrm{GSp}_{nb}(k')$ and, moreover, the projection map $\mathrm{GSp}_n(k') \to k'$ restricts to a surjective projection from H to k'. Since $\mathrm{GSp}_n(k)$ has a unique subgroup of index 2, we have $\mathrm{PGSp}_{n,k'}(k) \cong \mathrm{PSp}_n(k)$ if b is even and $\mathrm{PGSp}_n(k)$ if b is odd. It follows that $\mathrm{Aut}(\mathrm{PSp}_{nb}(k'))$ contains a subgroup isomorphic to $\mathrm{Aut}(\mathrm{PSp}_n(k))$ when b is odd and that, when b is even $\mathrm{PGSp}_{nb}(k')$ contains a subgroup $2 \times \mathrm{PSp}_n(k)$. Furthermore, by [10, 4.3.10] these subgroups are unique up to conjugacy. Thus we have the following lemma.

LEMMA 3.3: *If q is odd, then the inclusion of $\mathrm{PSp}_n(q^b)$ into $\mathrm{PSp}_{nb}(q)$ is a localization if and only if b is odd. In particular, for p odd and $a \neq 2^c$ with $c \geq 0$, $\mathrm{PSp}_4(p^a) \in \mathcal{L}_0$.* ∎

Notice that the previous lemma still leaves open the possibility that $\mathrm{PSp}_4(p) \in \mathcal{L}_0$ (the case $c = 0$).

LEMMA 3.4: *There are no localizations of $H = \mathrm{PSp}_4(q)$ with $q > 3$ into $\mathrm{Alt}(n)$.*

Proof: By [4] (see also [10, Table 5.2 a]) the minimal faithful permutation degree for H is $m = (q^4 - 1)/(q - 1)$. This is the index in H of both the maximal parabolic subgroups of H containing a fixed Borel subgroup of H. Since the maximal parabolic subgroups of H are not isomorphic to each other, H has two inequivalent permutation representations of degree m. Hence H is not localized in $\mathrm{Alt}(n)$ any n. ∎

Before continuing our investigations of localizations of $\mathrm{PSp}_4(q)$ we need some preparatory lemmas.

LEMMA 3.5: *Suppose that G acts absolutely irreducibly on the self-dual kG-module V by the representation ϕ. Let τ be the inverse transpose automorphism of $\mathrm{GL}(V)$. Then there exists $h \in \mathrm{GL}(V)$ such that $h\tau$ (the product of h and τ in $\mathrm{Aut}(\mathrm{GL}(V))$) centralizes $\phi(G)$. Furthermore, if $\phi(G) \leq \mathrm{GU}(V)$, then h can be chosen in $\mathrm{GU}(V)$.*

Proof: Let τ be the inverse transpose automorphism of $\mathrm{GL}(V)$. Since V is isomorphic to V^*, there exists $h \in \mathrm{GL}(V)$ such that $\phi(g)^h = \phi(g)^\tau$ for all $g \in G$. This means that $h\tau$ centralizes $\phi(G)$. Now suppose that $\phi(G) \leq \mathrm{GU}(V)$. Then by Lemma 2.5 there is a $h \in \mathrm{GU}(V)$ such that $\phi(g)^\tau = \phi(g)^h$ and again we are done. ∎

Suppose that K is a finite field, H is a finite group and V is an irreducible KH-module. Consider V as a $\mathrm{GF}(p)H$-module V_p. Then V_p is a homogeneous completely reducible module. Let W be an irreducible submodule of V_p and put $E = \mathrm{End}_{\mathrm{GF}(p)H}(W)$. Then E is called the **field of definition** of V.

LEMMA 3.6: *Suppose that p is a prime, K is a finite field of characteristic p, H is a finite group and V is an irreducible KH-module. Let E be the field of definition of V and F be the largest subfield of E which is also a subfield of K. Then there exists an irreducible FH-module W such that $V \cong W \otimes_F K$.*

Proof: Let $L = \mathrm{End}_{KH}(V)$. Then by [1, (25.10)]

$$V \otimes_K L = \bigoplus_{\sigma \in \mathrm{Gal}(L/K)} X^\sigma$$

where each X^σ is an absolutely irreducible LH-module and the number of irreducible submodules of $V \otimes_K L$ is $|\,\mathrm{Gal}(L/K)|$. Regarding V as an FH-module, V is a direct sum of isomorphic irreducible FH-modules. Let W be an irreducible submodule of the FH-module V. Then $E = \mathrm{End}_{FH}(W)$ and

$$W \otimes_F E = \bigoplus_{\tau \in \mathrm{Gal}(E/F)} Y^\tau$$

where each Y^τ is an absolutely irreducible EH-module and the number of irreducible submodules is $|\,\mathrm{Gal}(E/F)|$. Note that we have $W \otimes_F L \cong (W \otimes_F E) \otimes_E L$ and $W \otimes_F L \cong (W \otimes_F K) \otimes_K L$. Since each Y^τ in $W \otimes_F E$ is absolutely irreducible, we have

$$W \otimes_F L = \bigoplus_{\tau \in \mathrm{Gal}(E/F)} (Y^\tau \otimes_E L)$$

is a decomposition as a direct sum of irreducible LH-modules. Now V can be identified as an irreducible submodule of $W \otimes_F K$. Therefore, $V \otimes_K L$ is isomorphic to a submodule of $\bigoplus_{\tau \in \mathrm{Gal}(E/F)} (Y^\tau \otimes_E L)$. Since $|\,\mathrm{Gal}(E/F)| = |\,\mathrm{Gal}(L/K)|$, we conclude that $V \otimes_K L \cong W \otimes_F L$ and this means that $W \otimes_F K \cong V$ as claimed. ∎

LEMMA 3.7: *Suppose that p is an odd prime, $K = \mathrm{GF}(p^b)$, H is a finite group and V is an irreducible KH-module. Let $E = \mathrm{End}_{KH}(V)$. If V supports a non-trivial KH-invariant quadratic form and a non-trivial KH-invariant symplectic form, then $|E : K|$ is even, each irreducible EH-submodule of $V \otimes_K E$ is not self-dual and supports a non-degenerate unitary form. In particular, if $\dim_K V = n$, we have embeddings $H \leq U_{n/2}(p^b) \leq \mathrm{Sp}_n(p^b)$ and $H \leq U_{n/2}(p^b) \leq \mathrm{O}_n^\epsilon(p^b)$.*

Proof: Suppose that f is a non-trivial symplectic form on V, g a non-trivial quadratic form on V and suppose that they are both KH-invariant. Set $W = V \otimes_K E$. Define $\hat{f} = f \otimes E$ by

$$\hat{f}(v \otimes \lambda, w \otimes \mu) = f(v, w)\lambda\mu$$

for all $\lambda, \mu \in E$, and $v, w \in V$. Similarly set $\hat{g} = g \otimes E$. Then $\hat{f}$ is a non-degenerate symplectic form on W, $\hat{g}$ is a non-degenerate quadratic form on W and both are KH-invariant. In particular, W is a self-dual EH-module. By [1, (25.10)]

$$W = \bigoplus_{\sigma \in \mathrm{Gal}(E/K)} X^\sigma$$

where each X^σ is irreducible and $X^\sigma \cong X^\tau$ if and only if $\sigma = \tau$.

Since W is self-dual, X^* appears as one of the irreducible submodules in the decomposition of W above. Suppose that $X^* \cong X^\tau$ for some non-trivial $\tau \in \mathrm{Gal}(E/K)$. Since $X \cong X^{**} \cong (X^\tau)^* \cong (X^*)^\tau \cong X^{\tau^2}$, τ has order 2 and $[E : K]$ is even. Furthermore, in this case X supports an KH-invariant unitary form by [10, (2.10.15) (ii)] and this is part of our desired conclusion. Suppose that $X^* \cong X$. Then each X^σ is self-dual. We consider the forms $\hat{f}$ and $\hat{g}$ restricted to X. If for either form X is isotropic, then $V/X^\perp \cong X^*$ together with the decomposition of W given above, delivers a contradiction. Thus $\hat{f}$ and $\hat{g}$ both restrict to non-degenerate forms on X. But X is absolutely irreducible and so $\hat{f}$ and $\hat{g}$ are similar as E-bilinear forms by [1, Ex. 9.1(4)]. Of course this contradicts the fact that $\hat{f}$ is symplectic and $\hat{g}$ is quadratic. Therefore X is not self-dual and the proof is complete. ∎

LEMMA 3.8: *Assume that K is a field, H is a group and V is a KH-module which supports a non-degenerate symplectic form. Suppose that X is a self-dual irreducible KH-module and that X does not support a KH-invariant non-degenerate symplectic form. Then the multiplicity of X in the composition factors of V is even.*

Proof: Select a counterexample V of minimal dimension. Suppose that there is a non-zero KH-invariant isotropic subspace W of V. Then, as KH-modules, $V/W^\perp \cong W^*$ and so the multiplicity of X in the composition factors of the non-degenerate symplectic space $W^\perp/W$ is odd and we have a contradiction to our minimal choice of V. Thus every KH-invariant subspace of V is non-degenerate. Let U be a minimal KH-invariant subspace of V. Then U is non-degenerate and irreducible as a KH-module. As $V = U \perp U^\perp$, it follows that the multiplicity of X in at least one of the non-degenerate spaces U and $U^{\perp}$ is odd. Therefore, $V = U$ is irreducible. However, we then have $U \cong X$ which is a contradiction as X does not support a KH-invariant symplectic form. ∎

LEMMA 3.9: *Suppose that $H \cong \mathrm{Sp}_4(q)$ and k is an algebraically closed field of characteristic r with $(r,q) = 1$. Let V be a non-trivial kH-module of dimension less than $(q(q-1)^2)/2$. Then, if r is odd, V is one of the four Weil representations of dimension $(q^2 \pm 1)/2$, (two of dimension $(q^2-1)/2$ and two of dimension $(q^2+1)/2$) and, if r is even, then V is one of the two Weil modules of dimension $(q^2-1)/2$.*

Proof: This is a special case of [7, Theorem 2.1]. ∎

Before we investigate the possibility that H is localized in a group of Lie type, we record some facts about the Weil modules for $\mathrm{Sp}_{2n}(q)$.

LEMMA 3.10: *Suppose that $H \cong \mathrm{Sp}_{2n}(q)$ and k is an algebraically closed field of characteristic r with $(r,q) = 1$. If r is odd, let V_1, V_2, V_3 and V_4 be the four Weil modules for kH and choose notation so that $\dim V_1 = \dim V_2 = (q^n-1)/2$ and $\dim V_3 = \dim V_4 = (q^n+1)/2$. If r is even, let V_1 and V_2 be the two Weil modules for kH of dimension $(q^n-1)/2$. Then*

(1) *The pairs of modules V_1 and V_2, and, V_3 and V_4 are quasiequivalent by the diagonal automorphism of H.*
(2) *If $q \equiv 1 \pmod 4$ and r is odd, then V_1 and V_2 are faithful kH-modules while V_3 and V_4 are faithful $kH/Z(H)$-modules. All the Weil modules are self-dual, V_1 and V_2 support a kH-invariant symplectic form and V_3 and V_4 support a kH-invariant quadratic form.*
(3) *If $q \equiv -1 \pmod 4$ and r is odd, then V_3 and V_4 are faithful kH-modules while V_1 and V_2 are faithful $kH/Z(H)$-modules. The Weil modules are not self-dual; $V_1^* = V_2$ and $V_3^* = V_4$.*
(4) *If $r = 2$, then V_1 and V_2 are modules for $kH/Z(H)$ and V_1 and V_2 are self-dual if and only if $q \equiv 1 \pmod 4$. Moreover, V_1 and V_2 do not support a kH-invariant orthogonal form.*
(5) *Let z be a transvection (long root element) in H. If z and z^r are conjugate in H, then the field of definition of all the Weil modules is $\mathrm{GF}(r)$. If z are z^r are not conjugate then, the field of definition of all the Weil modules is $\mathrm{GF}(r^2)$.*

Proof: See [7, page 305]. ∎

We now return to localizations of $\mathrm{PSp}_4(q)$.

LEMMA 3.11: *Suppose that $H = \mathrm{PSp}_4(q)$ with $q = p^a \equiv -1 \pmod 4$, r is a prime which does not divide $|H|$ and $\mathrm{GF}(r)$ is the field of definition of the Weil representations of H (so take $r \equiv 1 \pmod{|G|}$ for example). Then the Weil representation ϕ of dimension $(q^2-1)/2$ followed by projection is a localization of H into $\mathrm{PSL}_{(q^2-1)/2}(r)$.*

Proof: Let $H = \mathrm{PSp}_4(q)$ and $G = \mathrm{GL}_{(q^2-1)/2}(r)$. By Lemma 3.9, $\phi(H)$ is a representative of the unique conjugacy class of subgroups of G isomorphic to H. By the choice of r, $C_G(\phi(H)) = Z(G)$ and, by Lemmas 3.10 (1) and 2.4, $N_G(\phi(H))/C_G(\phi(H))$ is isomorphic to a subgroup of index 2 in the automorphism group of H. Finally, by Lemma 3.10 (3), the two Weil representations of dimension $(q^2-1)/2$ are dual to each other and so, if τ represents the inverse transpose automorphism of G and $G_1 = G\langle\tau\rangle$, then $N_{G_1}(\phi(H))/C_{G_1}(\phi(H)) \cong \mathrm{Aut}(H)$. Thus $\mathrm{Aut}(H)$ embeds into $\mathrm{Aut}(\mathrm{PSL}_{(q^2-1)/2}(r))$ and we have fulfilled all the conditions of Theorem 1.2. This proves the lemma. ∎

Suppose that $p \equiv 1 \pmod 4$ and r is a prime which is not a quadratic residue modulo p. Then for a transvection (long root element) z in $\mathrm{PSp}_4(p)$, z and z^r are not conjugate in $\mathrm{PSp}_4(p)$. Thus by Lemma 3.9, the splitting field of the Weil representations is $\mathrm{GF}(r^2)$. Consider the embedding of $\mathrm{Sp}_4(p)$ into $\mathrm{GSp}_{(p^2-1)/2}(r^2)$. Then, as there are two quasiequivalence classes of Weil representations this extends to an embedding of $\mathrm{GSp}_4(p)$ into $\mathrm{GSp}_{(p^2-1)/2}(r^2).\langle\sigma\rangle$ where σ is a field automorphism of order 2. It follows that $\mathrm{PSp}_4(p)$ is localized in $\mathrm{PSp}_{(p^2-1)/2}(r^2)$. We record this result in the following lemma.

LEMMA 3.12: *If $p \equiv 1 \pmod 4$ and r is not a quadratic residue modulo p, then the Weil embedding of $\mathrm{PSp}_4(p)$ into $\mathrm{PSp}_{(p^2-1)/2}(r^2)$ is a localization.* ∎

It remains to deal with $\mathrm{PSp}_4(p^{2^c})$ with $c > 0$.

LEMMA 3.13: *Suppose that p is an odd prime and $H \cong \mathrm{PSp}_4(p^{2^c})$ with $c > 0$. If G is a simple classical group defined over a field of characteristic $r \neq p$, then there are no localizations of H into G.*

Proof: Let p, r, H, G and c be as in the statement of the lemma. Let z be a transvection (long root element) in H. Then, as $c \geq 1$, z is conjugate to z^r in H. Therefore, Lemma 3.10 (3), (4) and (5) indicate that the Weil modules are all self-dual and have field of definition $\mathrm{GF}(r)$. We set $q = p^{2^c}$ and let $m = (q^2-1)/2$ be the dimension of the smallest Weil modules. If r is odd, we set

$m^* = m+1$ and if r is even we put $m^* = m$. By Lemma 3.9, the only non-trivial irreducible representations of degree at most m^* are the Weil representations. In particular, there are no non-trivial representations of $\mathrm{Sp}_4(q)$ of dimension less than m. Throughout the proof of the lemma we shall exploit the Weil embeddings $\mathrm{PSp}_4(q) \leq \mathrm{O}_{m^*}(r) \leq \mathrm{GL}^{\epsilon}_{m^*}(r^b)$, $\mathrm{Sp}_4(q) \leq \mathrm{Sp}_m(r) \leq \mathrm{GL}^{\epsilon}_m(r^b)$ when r is odd, and $\mathrm{PSp}_4(q) \leq \mathrm{Sp}_m(r) \leq \mathrm{GL}^{\epsilon}_m(r^b)$ when $r = 2$. Aiming for a contradiction we suppose that H is localized in G and identify H with its image in G. In particular, this means that any subgroup of G which is isomorphic to H must in fact be conjugate to H. Our contradiction will come about by showing that either H has non-trivial centralizer in $\mathrm{Aut}(G)$ or that $\mathrm{Aut}(H)$ does not embed into $\mathrm{Aut}(G)$.

Suppose that $G = \mathrm{PSL}^{\epsilon}_n(r^b)$ and that $\hat{G} = \mathrm{GL}^{\epsilon}_n(r^b)$. Then $\hat{G}$ contains a subgroup $\hat{H}$ which projects to H in G and such that $\hat{H}$ is isomorphic to either $\mathrm{PSp}_4(q)$ or to $\mathrm{Sp}_4(q)$ and in the latter case the centre of $\hat{H}$ lies in the centre of $\hat{G}$. Plainly for H to be embedded in G, we must have $n \geq m$. If $n > m^*$, then the inclusions

$$\hat{H} = \mathrm{PSp}_4(q) \leq \mathrm{GL}^{\epsilon}_{m^*}(r^b) \leq \mathrm{GL}^{\epsilon}_{m^*}(r^b) \times \mathrm{GL}^{\epsilon}_{n-m^*}(r^b) \leq \mathrm{GL}^{\epsilon}_n(r^b)$$

show that H has non-trivial centralizer in $\mathrm{Aut}(G)$ unless $\mathrm{GL}^{\epsilon}_{n-m^*}(r^b) = \mathrm{GL}^{+}_1(2)$. In the former case we have a contradiction. In the latter case we note that $r^b = 2$ and that $m = m^*$. Hence we have the inclusions

$$\hat{H} = \mathrm{PSp}_4(q) \leq \mathrm{Sp}_m(2) \leq \mathrm{SL}_m(2) \leq \mathrm{SL}_{m+1}(2).$$

Now the graph automorphism of $\mathrm{SL}_{m+1}(2)$ normalizes the subgroup $\mathrm{SL}_m(2)$ and can be arranged to centralize the subgroup $\mathrm{Sp}_m(2)$ which is a contradiction. Thus $n \leq m^*$. If r is odd and $n = m^*$, then we have the inclusions

$$\hat{H} = \mathrm{PSp}_4(q) \leq \mathrm{O}_{m^*}(r) \leq \mathrm{O}_{m^*}(r^b) \leq \mathrm{GL}^{\epsilon}_{m^*}(r^b)$$

and again we have H centralized by a graph automorphism of G. Similarly, if $n = m$ we have

$$\hat{H} = \mathrm{Sp}_4(q) \leq \mathrm{Sp}_m(r) \leq \mathrm{Sp}_m(r^b) \leq \mathrm{GL}^{\epsilon}_m(r^b)$$

and H has non-trivial centralizer in $\mathrm{Aut}(G)$ once again. This shows that $G \not\cong \mathrm{PSL}^{\epsilon}_n(r^b)$.

Suppose that $G = \mathrm{P}\Omega^{\epsilon}_n(r^b)$ and that $\hat{G} = \mathrm{GO}^{\epsilon}_n(r^b)$ with $\hat{H}$ as before. Assume that r is odd and H is isomorphic to a subgroup of G. Then by Lemma 3.10 we require $n \geq m^*$. Notice that m^* is odd. If $n > m^*$, we have the inclusions

$$\hat{H} = \mathrm{PSp}_4(q) \leq \mathrm{O}_{m^*}(r) \leq \mathrm{O}_{m^*}(r^b) \leq \mathrm{O}_{m^*}(r^b) \times \mathrm{O}^{\epsilon}_{n-m^*}(r^b) \leq \mathrm{O}^{\epsilon}_n(r^b).$$

Since r is odd, $\mathrm{O}^{\epsilon}_{n-m^*}(r^b)$ is non-trivial and we infer that H has non-trivial centralizer in $\mathrm{PGO}^{\epsilon}_n(r^b)$, which is a contradiction. Hence $n = m^*$. Then $\hat{H} = \mathrm{PSp}_4(q) \leq \mathrm{O}_{m^*}(r) \leq \mathrm{O}_{m^*}(r^b)$. If $b > 1$, then a field automorphism of G centralizes H. So we must have $b = 1$. Then $\hat{H} = \mathrm{PSp}_4(q) \leq \mathrm{O}_{m^*}(r)$. Now by Lemma 3.10, H has two quasiequivalent orthogonal representations of dimension m^* which are inequivalent and so $\mathrm{Aut}(H)$ does not embed into $\mathrm{Aut}(G)$ by Lemma 2.4, and we have a contradiction (here we are using the fact that $\mathrm{Aut}(G) = \mathrm{PGO}_{m^*}(r)$). Now assume that $r = 2$. Then $m = m^*$, and when $n > m$, we have

$$\hat{H} = \mathrm{PSp}_4(q) \leq \mathrm{Sp}_m(2) \leq \mathrm{Sp}_m(2^a) \times 2 \leq \mathrm{O}^{\epsilon}_{m+2}(2^a) \leq \mathrm{O}^{\epsilon}_n(2^a)$$

and we see that H is centralized by an involution in G. By Lemma 3.9 there is no orthogonal representation of H of dimension m and so this concludes the case $G = \mathrm{PGO}^{\epsilon}_n(r^b)$.

Finally suppose that $G = \mathrm{PSp}_n(r^b)$ and that $\hat{G} = \mathrm{GSp}_n(r^b)$. If $n = m$, then the argument expounded above shows that $\hat{H}$ commutes with a field automorphism or that $\mathrm{Aut}(H)$ does not embed into $\mathrm{Aut}(G)$. Assume that r^b is odd. If $n \geq 2m^*$, then

$$\begin{aligned}\hat{H} = \mathrm{PSp}_4(q) &\leq \mathrm{O}_{m^*}(r) \leq \mathrm{GL}_{m^*}(r).2 \leq \mathrm{GL}_{m^*}(r^b).2 \\ &\leq \mathrm{Sp}_{2m^*}(r^b) \leq \mathrm{Sp}_n(r^b)\end{aligned}$$

and H has non-trivial centralizer in G, a contradiction. So $n < 2m^*$. As $q > 4$, and $n < 2m^*$, Lemma 3.9 implies the only non-trivial irreducible representations over a field of characteristic r are the Weil representations. So suppose that $\hat{H} \leq \mathrm{Sp}_n(r^a)$ and that V is the associated $\mathrm{GF}(r)\hat{H}$-module. Then, as n is even and $n \neq m$, V is not irreducible. If $\hat{H} \cong \mathrm{Sp}_4(q)$, then as $\hat{H}$ projects to $\mathrm{PSp}_4(q)$ in G, we must have that $Z(\hat{H}) \leq Z(\hat{G})$. Hence, in this case, every composition factor of $\hat{H}$ on V is a Weil module of dimension m. It follows that $n = 2m$ and $\hat{H} \cong \mathrm{Sp}_4(q)$ embeds as a diagonal subgroup of $\mathrm{Sp}_m(r) \times \mathrm{Sp}_m(r) \leq \mathrm{Sp}_{2m}(r) \leq \mathrm{Sp}_{2m}(r^b)$ and H is centralized by an involution in G, which is a contradiction. Thus $\hat{H} \cong \mathrm{PSp}_4(q)$ and, as $n < 2m^*$, V has one composition factor of dimension m^* and $n - m^*$ of dimension 1. By Lemma 3.10 (2), the Weil module of dimension m^* supports a quadratic form and not a symplectic form. Thus according to Lemma 3.8 this composition factor has to appear with multiplicity two and we have a contradiction as $n < 2m^*$. This contradiction concludes the investigation of the symplectic groups when r is odd.

Suppose that $r = 2$ and $n > m$. Then we have

$$\hat{H} = \mathrm{PSp}_4(q) \leq \mathrm{Sp}_m(2) \leq \mathrm{Sp}_m(2^b) \times \mathrm{Sp}_{n-m}(2^b) \leq \mathrm{Sp}_n(2^b)$$

and again H has non-trivial centralizer in G. Hence $G \not\cong \mathrm{PSp}_n(2^b)$ and we have our final contradiction. ∎

The next result is required when we consider the characteristic p representations of $\mathrm{Sp}_4(q)$.

LEMMA 3.14: *Suppose that $H = \mathrm{Sp}_4(p^a)$ and $K = \mathrm{GF}(p^b)$. Set $L = \mathrm{GF}(p^a)$ and let $F = \mathrm{GF}(p^d)$ be the largest field which is contained in both L and in K. Assume that V is an irreducible KH-module and $\dim_K V < 10\frac{a}{d}$. Then either*

(1) *$V \cong W \otimes_F K$ has dimension $4\frac{a}{d}$ or $5\frac{a}{d}$ where W is a natural 4-dimensional symplectic LH-module or a natural 5-dimensional orthogonal LH-module considered as an FH-module; or*

(2) *a is even, $V \cong W \otimes_F K$ has dimension $8\frac{a}{d}$ with W isomorphic to $M \otimes M^\sigma$ considered as an FH-module where M is a natural 4-dimensional symplectic LH-module and σ is the field automorphism of L of order 2. In particular, W supports a non-degenerate KH-invariant quadratic form.*

Proof: Let $L^* = \mathrm{GF}(p^e)$ be the field of definition of V and k be an algebraically closed field of characteristic p. Then by [10, (5.4.6)(i)] we see that e divides a. Let $F_0 = \mathrm{GF}(p^f)$ be the largest subfield of K contained in L^*. By Lemma 3.6, there exists an irreducible F_0H-module W such that $V \cong W \otimes_{F_0} K$. In particular, $\dim_K V = \dim_F W \otimes_{F_0} F = \dim_{F_0} W$. Now $W \otimes_{F_0} k = \bigoplus_{\tau \in \mathrm{Gal}(L^*/F_0)} Y^\tau$ where, by the Steinberg Tensor Product Theorem [10, (5.4.6)(i)], $Y = \bigotimes_{\sigma \in \mathrm{Gal}(L/L^*)} M^\sigma$ for some irreducible kH-module M. Therefore, $\dim_{F_0} W = |\mathrm{Gal}(L^*/F_0)|(\dim_k M)^{a/e}$ for some irreducible kH-module M. Now $|\mathrm{Gal}(L^*/F_0)| = e/f$. Therefore, as $\dim_k M \geq 4$ for all irreducible kH-modules M, we have

$$4^{\frac{a}{e}}\frac{e}{f} \leq \dim_K V < 10\frac{a}{d} \leq 10\frac{a}{f}$$

and so $\frac{a}{e} < 3$. If $a = e$, the claim in (1) follows from [16, Table 6.22]. If $\frac{a}{e} = 2$, then we have $(\dim_k M)^2 \frac{e}{f} < 10\frac{a}{d}$ which means that $(\dim_k M)^2 < 20\frac{f}{d}$. It follows that $\dim_k M = 4$ and $f = d$. Thus the claim in (2) holds. ∎

LEMMA 3.15: *Suppose that $H \cong \mathrm{PSp}_4(p^a)$ with $a = 2^c$ and $c > 0$. If H is localized in $G \cong \mathrm{PSL}_n^\epsilon(p^b)$ or $\mathrm{P\Omega}_n^\epsilon(p^b)$, then $n = 5$, $a = b$ and $G \cong \mathrm{P\Omega}_5(p^a) \cong H$.*

Proof: Suppose that H is localized in G. Let $\hat{G}$ be one of $\mathrm{GL}_n^\epsilon(p^b)$ or $\mathrm{GO}_n^\epsilon(p^b)$ and let $\hat{H}$ be the derived subgroup of the preimage of H in $\hat{G}$. So $\hat{H}$ is isomorphic to either $\mathrm{PSp}_4(p^a)$ or $\mathrm{Sp}_4(p^a)$. Let $\mathrm{GF}(p^d)$ be the largest subfield of

$\mathrm{GF}(p^a)$ which is also a subfield of $\mathrm{GF}(p^b)$. Suppose first that $n \geq 5\frac{a}{d}$; then the embeddings

$$\hat{H} = \mathrm{PSp}_4(p^a) \leq \mathrm{O}^{\theta}_{5\frac{a}{d}}(p^d) \leq \mathrm{O}^{\theta}_{5\frac{a}{d}}(p^b) \times \mathrm{O}^{\mu}_{n-5\frac{a}{d}}(p^b) \leq \mathrm{O}^{\epsilon}_n(p^b) \leq \mathrm{GL}^{\gamma}_n(p^b)$$

show that if $n > 5\frac{a}{d}$, then H has non-trivial centralizer in $\mathrm{Aut}(G)$. If $n = 5\frac{a}{d}$, then in the linear and unitary case an outer automorphism of G centralizes H. In the orthogonal case we have

$$\hat{H} = \mathrm{PSp}_4(p^a) \leq \mathrm{O}^{\epsilon}_{5\frac{a}{d}}(p^d) \leq \mathrm{O}^{\epsilon}_{5\frac{a}{d}}(p^b).$$

If $d < b$, then a field automorphism of G centralizes H. So $d = b$. Assume that $d < a$. Then, as a is a power of 2,

$$\hat{H} = \mathrm{PSp}_4(p^a) = \mathrm{O}_5(p^a) \leq \mathrm{O}^{\epsilon}_{10}(p^{\frac{a}{2}}) \leq \mathrm{O}^{\epsilon}_{5\frac{a}{b}}(p^b)$$

and the intermediate embedding $\mathrm{O}_5(p^a) \leq \mathrm{O}^{\epsilon}_{10}(p^{\frac{a}{2}})$ shows that H has non-trivial centralizer in $\mathrm{Aut}(G)$ (see [6, Table 4.5.1]). The only other possibility is that $a = b$ and $H \cong G$ which is our desired conclusion.

We now suppose that $n < 5\frac{a}{d}$. It follows from Lemma 3.14 that the only non-trivial composition factor of V has dimension $4\frac{a}{d}$ and results from an embedding of $\hat{H} \cong \mathrm{Sp}_4(p^a)$ into $\hat{G}$. Since this embedding leads to an embedding of H into G, we must have $Z(\hat{G}) \geq Z(\hat{H})$. Thus $n = 4\frac{a}{d}$. If $\hat{G} = \mathrm{GL}^{\epsilon}_{4\frac{a}{d}}(p^b)$, then H has non-trivial centralizer in $\mathrm{Aut}(G)$ by Lemma 3.5. Thus $G = \mathrm{O}^{\epsilon}_{4\frac{a}{d}}(p^b)$. But then H supports both a non-trivial quadratic and symplectic form and as every absolutely irreducible module for $\mathrm{Sp}_4(p^a)$ is self-dual (see [10, (4.5.3)]), Lemma 3.7 delivers a contradiction. This completes the proof of the lemma. ∎

The proof of the next result mirrors the proof of Lemma 3.15 but is slightly more technical.

LEMMA 3.16: *Suppose that $H \cong \mathrm{PSp}_4(p^a)$ with $a = 2^c$ and $c > 0$. If H is localized in $G \cong \mathrm{PSp}_n(p^b)$, then $n = 4$, $a = b$ and $G \cong \mathrm{PSp}_4(p^a) \cong H$.*

Proof: Suppose that H is localized in G. Let $\hat{G} = \mathrm{Sp}_n(p^b)$ and V be the associated $\mathrm{GF}(p^b)H$-module. As in Lemma 3.15 let $\hat{H}$ be the derived subgroup of the preimage of H in $\hat{G}$. So $\hat{H}$ is isomorphic to either $\mathrm{PSp}_4(p^a)$ or $\mathrm{Sp}_4(p^a)$. We let $\mathrm{GF}(p^d)$ be the largest subfield of $\mathrm{GF}(p^a)$ which is also a subfield of $\mathrm{GF}(p^b)$. Suppose first that $n \geq 10\frac{a}{d}$; then

$$\begin{aligned}\hat{H} = \mathrm{PSp}_4(p^a) &\leq \mathrm{O}^{\epsilon}_{5\frac{a}{d}}(p^d) \leq \mathrm{O}^{\epsilon}_{5\frac{a}{d}}(p^b) \leq \mathrm{GL}_{5\frac{a}{d}}(p^b).2 \\ &\leq \mathrm{Sp}_{10\frac{a}{d}}(p^b) \leq \mathrm{Sp}_{10\frac{a}{d}}(p^b) \times \mathrm{Sp}_{n-10\frac{a}{d}}(p^b) \leq \mathrm{Sp}_n(p^b).\end{aligned}$$

So we see that an outer element of $\mathrm{GL}_{5\frac{a}{d}}(p^b).2$ centralizes $\hat{H}$ and we have a contradiction. Therefore, $n < 10\frac{a}{d}$. Suppose that $\hat{H}$ acts irreducibly on V. Then the possibilities for V are given in Lemma 3.14. So $n = 4\frac{a}{d}$, $5\frac{a}{d}$ or $8\frac{a}{d}$. In the first case we have

$$\hat{H} = \mathrm{Sp}_4(p^a) \leq \mathrm{Sp}_{4\frac{a}{d}}(p^d) \leq \mathrm{Sp}_{4\frac{a}{d}}(p^b).$$

If $b > d$, then H is centralized by a field automorphism of G. Thus $b = d$. If $b < a$, then, as $a = 2^c$, the inclusion $\hat{H} = \mathrm{Sp}_4(p^a) \leq \mathrm{Sp}_8(p^{\frac{a}{2}}) \leq \mathrm{Sp}_{4\frac{a}{b}}(p^b)$ shows that H is centralized by a diagonal automorphism of G (see [6, Table 4.5.1, pg. 172]) which is a contradiction. Therefore, $b = a$ in which case $H \cong G$.

Suppose that $n = 5\frac{a}{d}$. Then $\hat{H} = \mathrm{PSp}_4(p^a) \leq \mathrm{O}_{5\frac{a}{d}}(p^d) \leq \mathrm{O}_{5\frac{a}{d}}(p^b)$ as well as $\hat{H} \leq \mathrm{Sp}_n(p^b)$. Therefore, $\hat{H}$ supports both a symplectic and quadratic form on V and as all the absolutely irreducible modules for $\mathrm{Sp}_4(p^a)$ are self-dual we have a contradiction to Lemma 3.7.

If $n = 8\frac{a}{d}$ and $\hat{H}$ is irreducible, then there are at least two non-conjugate embeddings of H into G (the irreducible one just mentioned and the reducible one resulting from the diagonal embedding $\mathrm{Sp}_4(p^a) \leq \mathrm{Sp}_{4\frac{a}{d}}(p^b) \leq \mathrm{Sp}_{4\frac{a}{d}}(p^b) \times \mathrm{Sp}_{4\frac{a}{d}}(p^b)$).

Thus $\hat{H}$ is not an irreducible subgroup of $\hat{G}$. If $\hat{H} \cong \mathrm{Sp}_4(p^a)$, then every composition factor of V must be faithful for $\hat{H}$. Since $n < 10\frac{a}{d}$ the only way this can happen is if $n = 8\frac{a}{d}$ and $\hat{H}$ embeds diagonally into $\mathrm{Sp}_{4\frac{a}{d}}(p^b) \times \mathrm{Sp}_{4\frac{a}{d}}(p^b)$. But in this case H centralizes an involution in G. So $\hat{H} \cong \mathrm{PSp}_4(p^a)$ and, as $n < 10\frac{a}{d}$, Lemma 3.14 implies $\hat{H}$ has just one non-trivial composition factor on the natural $\hat{G}$-module V. In particular, $\hat{H}$ must fix a 1-dimensional subspace of V. But then H centralizes a long root subgroup (transvection group) of G and this is our final contradiction. ∎

We now begin our investigation of localizations of $\mathrm{PSp}_4(p^a)$ into the exceptional groups. The proof of the next lemma is taken from parts of [3, 2.13].

LEMMA 3.17: *Suppose that $H = 2^4.\mathrm{Alt}(5)$ and $C_H(O_2(H)) = O_2(H)$. Then H is not isomorphic to a subgroup of $\mathrm{F}_4(s)$ for any odd prime power s.*

Proof: Aiming for a contradiction we identify H with a subgroup of $G = \mathrm{F}_4(s)$. Let $Q = O_2(H)$. By [6, Table 4.5.1], G has two conjugacy classes of involutions. Let $x, y \in G$ be representatives of these classes and assume that $C_G(x) \cong (\mathrm{SL}_2(s) \circ \mathrm{Sp}_6(s)).2$ and $C_G(y) \cong \mathrm{Spin}_9(s)$. Suppose that $x \in Q$. Let C be the component of $C_G(x)$ with $C \cong \mathrm{Sp}_6(s)$. Then, as Q has order 2^4, and the 2-rank of $\mathrm{PGL}_2(s)$ is 2, we have $|Q \cap C| \geq 2^2$. Let $h \in (Q \cap C) \setminus \{x\}$. Then on the natural 6-dimensional $\mathrm{Sp}_6(s)$-module, by changing h to hx if necessary, we may assume that h has four eigenvalues equal to -1. Therefore, $C_C(h)/\langle h\rangle \cong \mathrm{SL}_2(s)\times \mathrm{PSp}_4(s)\cong \mathrm{SL}_2(s)\times\Omega_5(s)$. Since $\mathrm{PSp}_6(s)$ does not contain a subgroup isomorphic to $\Omega_5(s)$, we infer that h is not conjugate to x. Therefore, $y^G\cap Q$ is non-empty and so we may assume that $y\in Q$. Now $Q\le C_G(y)\cong \mathrm{Spin}_9(s)$. When we identify $\mathrm{Spin}_9(s)/\langle y\rangle$ with $\Omega_9(s)$, the involutions $\mathrm{diag}(-1^2,1^7)$ and $\mathrm{diag}(-1^6,1^3)$ have preimages of order 4 in $C_G(y)$ and the elements $\mathrm{diag}(-1^4,1^5)$ and $\mathrm{diag}(-1^8,1)$ have preimages which are involutions in $C_G(y)$. Moreover, two involutions which are conjugate to $\mathrm{diag}(-1^8,1)$ and which commute in $C_G(y)/\langle y\rangle$ have product of order 4 (their product centralizes a codimension 2-subspace). Thus $Q\setminus\langle y\rangle$ consists of involutions all of which are conjugate to $\mathrm{diag}(-1^4,1^5)$. Let $q_1,q_2\in Q\setminus\langle y\rangle$ with $q_1q_2=y$. Then $O^2(C_{C_G(y)}/\langle y\rangle)\cong \Omega_4^+(s)\times\Omega_5(s)$ and so, for $i=1,2$, we infer that $C_G(q_i)/\langle q_i\rangle$ contains a subgroup isomorphic to $(2.\Omega_5(s)\times \mathrm{SL}_2(s))\circ \mathrm{SL}_2(s)$. There is no such subgroup in $\Omega_9(s)$ and so $q_1,q_2\in x^G$. Hence y is the unique conjugate of y contained Q and so $\langle y\rangle$ is normal in H and this is a contradiction. ∎

We now deduce the following interesting result which will be useful for us in the case when $r=p$.

Lemma 3.18: *Suppose that p and r are odd prime numbers and b is a positive integer. Then the simple groups $\mathrm{F}_4(r^b)$, $\mathrm{G}_2(r^b)$, ${}^3\mathrm{D}_4(r^b)$, ${}^2\mathrm{G}_2(3^{2b+1})$ with $r=3$ and ${}^2\mathrm{F}_4(2^{2b+1})$ with $r=2$ do not contain a subgroup isomorphic to $\mathrm{PSp}_4(p)$.*

Proof: Because $\mathrm{PSp}_4(p)$ contains a subgroup isomorphic to $2^4.\mathrm{Alt}(5)$, $\mathrm{PSp}_4(p)$ is not isomorphic to a subgroup of $\mathrm{F}_4(r^b)$ by Lemma 3.17. As $\mathrm{G}_2(r^b)$ and ${}^3\mathrm{D}_4(r^b)$ are isomorphic to subgroups of $\mathrm{F}_4(r^b)$ we have the result for these groups. The observation that ${}^2\mathrm{G}_2(3^{2b+1})$ is contained in $\mathrm{G}_2(3^{2b+1})$ and ${}^2\mathrm{F}_4(2^{2b+1})$ is contained in $\mathrm{F}_4(2^{2b+1})$ finishes the lemma. ∎

Lemma 3.19: *Suppose that p is an odd prime and $G=\mathrm{E}_8(p^{2^bd})$ where d is odd. If G contains a subgroup H isomorphic to $\mathrm{PSp}_4(p^{2^c})$ or $\mathrm{Sp}_4(p^{2^c})$, then $b+1\ge c$.*

Proof: Let T be the cyclic subgroup of H of order $(p^{2^{c+1}}+1)/2$. Then $|T|$ is divisible by a prime l which is a primitive prime divisor of $p^{2^{c+2}}-1$. Notice that if l is a divisor of p^d-1, then 2^{c+2} divides d. On the other hand, setting $t=p^{2^bd}$, we have that

$$|\mathrm{E}_8(t)|=t^{120}(t^{30}-1)(t^{24}-1)(t^{20}-1)(t^{18}-1)(t^{14}-1)(t^{12}-1)(t^8-1)(t^2-1).$$

It follows that for $\mathrm{PSp}_4(p^{2^c})$ to be a subgroup of $\mathrm{E}_8(p^{2^b d})$ we must have $2^{b+3} \geq 2^{c+2}$. Hence $b+1 \geq c$. ∎

PROPOSITION 3.20: *Suppose that p is an odd prime. Then there are no localizations of $\mathrm{PSp}_4(p^{2^c})$ with $c > 0$ into an exceptional group of Lie type defined over a field of characteristic p.*

Proof: The result follows trivially from Lemma 3.18 for all the groups other than $\mathrm{E}_n(p^k)$, $n \in \{6,7,8\}$ and ${}^2\mathrm{E}_6(p^k)$. So assume that $H \cong \mathrm{PSp}_4(p^{2^c})$ is localized in one of the aforementioned groups. Then, writing $k = 2^b d$, we have $b+1 \geq c$ by Lemma 3.19. If $b \geq c$, then, for $\epsilon = \pm$, we have the containments

$$\mathrm{E}_6^\epsilon(p^k) \geq \mathrm{E}_6^\epsilon(p^{2^b}) \geq \mathrm{SL}_6^\epsilon(p^{2^b})/(p^{2^b} - \epsilon, 3) \geq \Omega_5(p^{2^b}) \geq \Omega_5(p^{2^c})$$

and we see a subgroup isomorphic to H centralizing an involution in $\mathrm{E}_6^\epsilon(p^k)$, which is a contradiction. Since the universal version of $\mathrm{E}_6(p^k)$ is contained in $\mathrm{E}_n(p^k)$ for $n = 7, 8$, we have a contradiction. Therefore, $c = b+1$. In this case we use the containments

$$\mathrm{E}_6^\epsilon(p^k) \geq \mathrm{E}_6^\epsilon(p^{2^b}) \geq \mathrm{PSp}_8(p^{2^b}) \geq \mathrm{PSp}_4((p^{2^b})^2) = \mathrm{PSp}_4(p^{2^c})$$

and observe that $\mathrm{PSp}_8(p^{2^b})$ is contained in the centralizer of a graph automorphism of $\mathrm{E}_6^\epsilon(p^{2^b})$ [6, Table 4.5.1]. Finally, noting once again that the universal version of $\mathrm{E}_6(p^k)$ is contained in $\mathrm{E}_n(p^k)$ for $n = 7, 8$, and this subgroup centralizes semisimple elements from a maximal torus we have shown that H is not localized in any exceptional group and we are done. ∎

The case of cross characteristic embeddings of $\mathrm{PSp}_4(q)$ into exceptional groups is far more straightforward. We deal explicitly with localizations of H into $\mathrm{E}_8(r^b)$.

LEMMA 3.21: *Suppose that p is an odd prime, $q = p^{2^c}$ with $c > 0$ and $H = \mathrm{PSp}_4(q)$. Then H is not isomorphic to a subgroup of $\mathrm{E}_8(r^b)$ for any prime $r \neq p$.*

Proof: Let $G = \mathrm{E}_8(r^b)$. Using Lemma 3.9 we have that $(q^2-1)/2 \leq 248$. Thus as $q = p^{2^c}$ with $c > 0$, $q = 9$. However, this solitary possibility is explicitly ruled out in [14, Proposition 8.1]. ∎

We present a rather weak result about sporadic simple groups.

LEMMA 3.22: *Suppose that p is an odd prime, $q = p^{2^c}$ with $c > 0$ and $H = \mathrm{PSp}_4(q)$. Then H is not isomorphic to a subgroup of a sporadic simple group.*

Proof: Let S be a Sylow p-subgroup of H. Then $|S| = q^4 \geq p^4$. By considering the orders of the Sylow p-subgroups of the sporadic simple groups and recalling that $q = p^{2^c} \equiv 1 \pmod 4$, we have that $p^{2^c} \in \{3^2, 3^4, 5^2\}$. If $q = 3^4$ or $q = 5^2$, $|H|$ is divisible by the primes 193 and 313 respectively. Since these primes do not divide the orders of any sporadic simple group, we get $p^{2^c} = 3^2$. However, then 41 divides $|H|$ and this means that the only possibility is that it is a subgroup of the Monster simple group M and for this case we refer to [19]. ∎

Finally we assemble the pieces to provide a proof of Proposition 3.1. First of all Lemmas 3.3 and 3.11, 3.12 prove Proposition 3.1(1). So suppose that $q = p^{2^c}$ with $c > 0$. Then Lemmas 3.2, 3.4, 3.13, 3.15, 3.16, 3.20, 3.21 and 3.22 prove Proposition 3.1(2). Of course Propositions 2.9 and 3.1 together provide the proof of Theorem 1.1.

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